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BEAM CURRENT LIMITATIONS DUE TO INSTABILITIES IN MODIFIED AND CONVENTIONAL BETATRONS

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Current thresholds for longitudinal and transverse instabilities are calculated for beams of specified dimensions in conventional ($B_\theta = 0$) and modified ($B_\theta \neq 0$) betatrons, using simple models for the longitudinal and transverse impedances. Self-field effects of the beam are included and lead to a novel, competitive effect between the stabilization mechanism and instability growth. This competition results in a multi-valuedness in the limiting current vs beam energy spread plot, even for conventional betatrons. Accessibility of the various limiting current levels appears to depend upon the rate of beam injection. The stabilizing effects of betatron oscillations are discussed and written as the sum of three physically interpretable contributions to an effective energy spread. We find that the presence of a strong toroidal field can significantly improve the current carrying capacity of the accelerator.

I. INTRODUCTION

The addition of a toroidal magnetic field to a conventional betatron has been shown theoretically[1–5] to increase the equilibrium current that may be confined by a factor $1/2(B_\theta/B_z)^2$, for large values of B_θ/B_z. In conventional accelerators without solenoidal focusing, however, beam stability considerations place the actual limit on beam current.[6–8] Therefore it becomes important to analyze the stability conditions and associated limiting currents for a given beam equilibrium in the presence of a toroidal field. In this paper we present such an analysis for both longitudinal and transverse modes.

A device in which a toroidal magnetic field is superimposed on the usual weak focusing betatron field has come to be called a "Modified Betatron". See Fig. 1. A stability analysis of this accelerator necessarily must include the strong self and induced (wall image) fields of the electron beam. It is primarily the inclusion of these fields that distinguishes this work from the stability analysis performed[9] for the so-called plasma betatron in which self fields are much less important. These self field effects, however, will be seen to have a dramatic effect on the current versus energy spread scaling; namely we predict the existence of more than one stable value of current

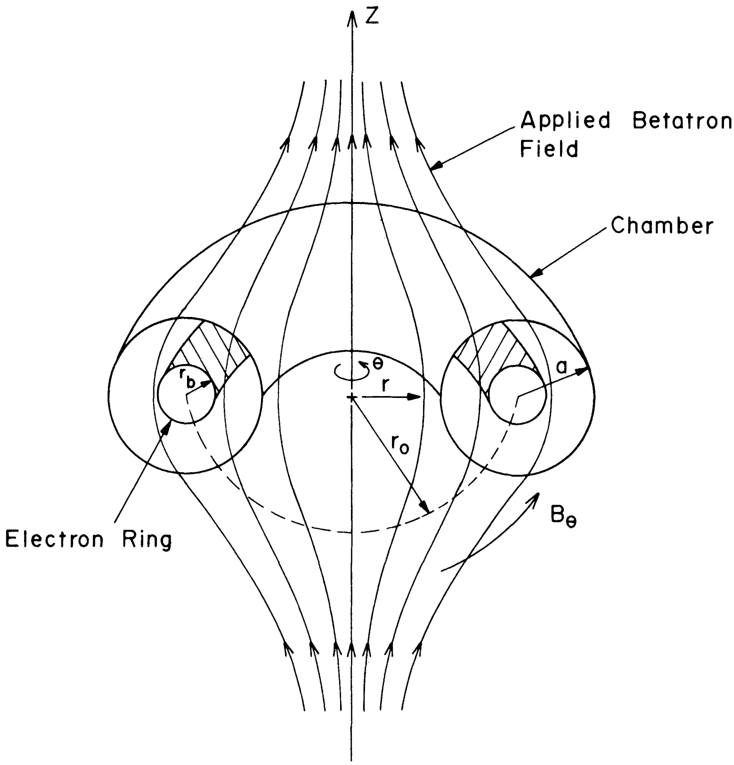

FIGURE 1 Modified Betatron configuration.

for a given beam energy spread. This somewhat surprising result will be discussed later. The central result of this work, however, is that significantly more current may be carried by a beam in a modified betatron configuration than in a conventional betatron of the same dimensions, assuming equal beam sizes and energy spreads. In reaching this conclusion we have included not only the effects of self fields, but also the (stabilizing) effects of betatron oscillations and the (destabilizing) effects of short wavelength enhancements to the longitudinal and transverse impedances due to chamber resonances. Below we discuss the dispersion relation for arbitrary toroidal fields and currents, describe our model for the impedances, and present analytical and numerical results from the dispersion relation.

II. DISCUSSION

A dispersion relationship for both the longitudinal and transverse modes in a modified betatron accelerator configuration has been derived.[10,10a] Included in the derivation are beam self-field effects, induced-field effects arising from wall-image charges and currents, as well as finite chamber wall conductivity effects. Toroidal corrections to the equilibrium beam self fields and chamber wall image fields have been neglected. The longitudinal and transverse impedances, which characterize the beam environment, will here be incorporated in a phenomenological way in the short wavelength limit. The dispersion relation, therefore, treats disturbances of all wavelengths, including wavelengths much longer or shorter than the chamber minor radius.

The short wavelength model for the impedances contains effects associated with propagating chamber modes. These effects can significantly affect the instability growth rates. With the inclusion of the short wavelength contributions to the impedances, a realistic estimate for the current limitations due to the various instabilities can be obtained for the modified betatron configuration and compared with those of a conventional betatron. To perform a meaningful comparison we will choose identical parameters for the two types of betatrons, i.e., same geometry, injection energy, field index, etc. The only difference of course will be that the modified betatron configuration will include a toroidal magnetic field.

The dispersion relation[10] for the longitudinal and transverse modes of a cold beam may be written

$$1 = \frac{\tilde{\omega}_{\parallel}^2}{\Delta\tilde{\omega}_l^2}\left\{\frac{1}{\gamma^2} + \frac{(\Delta\tilde{\omega}_l^2 - \tilde{\omega}_{\perp}^2)}{(\Delta\tilde{\omega}_l^2 - \tilde{\omega}_{\perp}^2)^2 - b^2\Delta\tilde{\omega}_l^2}\right\}, \tag{1}$$

where $\Delta\tilde{\omega}_l = (\omega - l\omega_c)/\omega_c$, ω is the complex mode frequency, $l = 1, 2, 3, \ldots$ is the longitudinal (toroidal) harmonic mode number, $\omega_c = \Omega_z/\gamma$ is the electron rotation frequency, $\Omega_z = |e|B_z/m_0c$ is the non-relativistic cyclotron frequency, $\gamma = (1 - v_0^2/c^2)^{-1/2}$, v_0 is the longitudinal (toroidal) beam velocity component, $\tilde{\omega}_{\parallel}^2 = -2il^2(\nu/\gamma)(Z_{\parallel}/lZ_0)$, ν is Budker's parameter ($\nu = I[A]/17 \times 10^3$, $Z_{\parallel} = Z_{\parallel}(\omega)$ is the total effective longitudinal impedance, $Z_0 = 4\pi/c$ ($Z_0 = 377\ \Omega$ in MKS units), $\tilde{\omega}_{\perp}^2 = 1/2 - 2(\nu/\gamma^3)(r_0/r_b)^2 - 2i(\nu/\gamma)(r_0Z_{\perp}/Z_0)$, r_0 is the major electron beam radius, r_b is the minor electron beam radius, $Z_{\perp} = Z_{\perp}(\omega)$ is the total effective transverse impedance, $b = B_\theta/B_z$ and B_θ is the toroidal magnetic field. In (1) finite amplitude betatron oscillations were neglected, the external field index was taken to be 1/2 and the electron beam was assumed to be mono-energetic, highly relativistic ($v_0 \approx c$) and circular in cross section.

III. APPROXIMATE REPRESENTATION OF IMPEDANCES

In our model the longitudinal impedance $Z_{\parallel}(\omega)$ is taken to consist of three terms

$$Z_{\parallel} = Z_{\parallel,s} + Z_{\parallel,\sigma} \times Z_{\parallel,r}. \tag{2}$$

The first term in (2) is the long wavelength space charge shielding contribution associated with a smooth infinitely conducting chamber and is given by

$$Z_{\parallel,s} = lZ_0(i/2\gamma_w^2)(1 + 2\ln a/r_b), \tag{2a}$$

where γ_w is the relativistic factor corresponding to the wave phase velocity and a is the minor radius of the toroidal chamber. Due to the $1/\gamma_w^2$ factor this impedance term is typically quite small; it may even change sign.

The part of the longitudinal impedance due to the resistive nature of the chamber wall, in the long wavelength regime, is

$$Z_{\parallel,\sigma} = lZ_0(1 - i)\delta/2a. \tag{2b}$$

where $\delta = c(2\pi\sigma|\omega|)^{-1/2}$ is the skin depth and σ is the wall conductivity. It has been assumed in (2b) that the skin depth is small compared with the thickness of the chamber wall.

Finally, the last term in (2) represents the resonance contribution to the total longitudinal impedance and arises from the fact that the chamber can support propagating waves. To obtain the exact form for $Z_{\|,r}$ would require a rather involved analysis of the beam-chamber structure and is beyond the goals of the present paper. We will, therefore, simply represent this contribution by the phenomenological expression[11]

$$Z_{\|,r} = R_0 \frac{\omega}{\omega_r}\left(\frac{\omega/\omega_r - iQ(1 - (\omega/\omega_r)^2)}{(\omega/\omega_r)^2 + Q^2(1 - (\omega/\omega_r)^2)^2}\right), \tag{2c}$$

where R_0 defines the chamber shunt impedance, $\omega_r \approx 2.4\, c/a$ is the cutoff frequency of the lowest order chamber mode, and Q is the quality factor associated with the chamber. The chamber shunt impedance, R_0, can be estimated by noting that near a resonant frequency $\omega = \omega_r$ the longitudinal impedance is roughly equal to the free space impedance

$$Z_{\text{f.s.}} \approx \frac{Z_0}{2}(\sqrt{3} - i)l^{1/3}. \tag{3}$$

It follows, therefore, that $R_0 \approx Z_0 l_r^{1/3}$ where l_r is the toroidal mode number associated with the resonant frequency, i.e., $l_r = \omega_r/\omega_c$.

The expression for the total transverse impedance $Z_\perp(\omega)$ is also written as the sum of three contributions

$$Z_\perp = Z_{\perp,s} + Z_{\perp,\sigma} + Z_{\perp,r}. \tag{4}$$

The long wavelength space charge part of the impedance, for a smooth infinitely conducting chamber, is

$$Z_{\perp,s} = ir_0 Z_0(1/r_b^2 - 1/a^2)/\gamma^2. \tag{4a}$$

The second term is the long wavelength contribution due to the resistive nature of the wall and is given by

$$Z_{\perp,\sigma} = r_0 Z_0(1 - i)\delta/a^3. \tag{4b}$$

Chamber resonances contribute to the transverse impedance a part which we will represent by the form[11]

$$Z_{\perp,r} = \frac{2r_0}{la^2} Z_{\|,r}, \tag{4c}$$

where $Z_{\|,r}$ is defined in (2c).

In the present work, we will not be concerned with resistive wall effects which lead, in conventional accelerators, to well known longitudinal and transverse instabilities[6] having, however, comparatively slow growth rates. Equations (2b) and (4b) are included here only for reference purposes.

We remark that with the impedances as defined above the dispersion relation is virtually independent of the beam minor radius, r_b, except for the weak logarithmic dependence in (2a). This is reasonable if we think of the dynamics involved in the various instabilities, that is, if we recall that the beam centroid moves transversely, even in the "longitudinal" or negative mass instability in which beam bunching occurs as the beam centroid moves in or out radially. Motion of the beam centroid is affected by the externally applied fields, including those due to wall image charges and currents; these fields, unlike beam self fields which are carried along transversely by the moving beam, do not depend on the minor radius of the beam.

IV. STABILITY CONDITION AND LIMITING CURRENT

To obtain the limiting current, based on stability requirements, for the modified and conventional betatron a stability criterion is needed. If the distribution in particle rotation frequencies is Lorentzian in shape the criterion for stability is simply

$$\Gamma \leqslant l|\Delta\Omega|, \tag{5}$$

where Γ is the growth rate in the absence of a frequency spread and $\Delta\Omega$ is the half width of the frequency spread on the beam. It should be noted that the large tails associated with a Lorentzian distribution make the criterion in (5) somewhat less stringent than would be the case if a more realistic choice of frequency distribution were used. However, the use of a more realistic distribution of particle frequencies would result in a considerably more involved stability criterion. Since we are interested here mainly in making a comparison of the limiting currents of the modified and conventional betatrons, consistent use of a Lorentzian for both devices should serve our purpose.

Both an intrinsic longitudinal energy spread of the beam electrons as well as finite-amplitude betatron oscillations will produce a spread in revolution frequencies. Solving the particle-orbit equations correct to second order in the betatron-oscillation amplitude, with self-field effects and intrinsic energy spread effects included, the average spread in the beam revolution frequency is found to be given by

$$|\Delta\Omega| = |\Delta\Omega|_{\Delta E} + |\Delta\Omega|_B, \tag{6}$$

where

$$|\Delta\Omega|_{\Delta E} = \frac{1}{2}\,\omega_c|\alpha|(\Delta E/E) \tag{6a}$$

$$|\Delta\Omega|_B = \omega_c(r_b/2r_0)^2|b^2/2 - n_s - 3/2|, \tag{6b}$$

with $\alpha = (1/2 - n_s)^{-1} - \gamma^{-2}$. The two terms on the right hand side of (6) are due to an intrinsic energy spread and to finite amplitude betatron oscillations, respectively. In (6a) the fractional intrinsic energy spread is denoted by $\Delta E/E$ where $E = \gamma m_0 c^2$ is the beam particle energy, $(\gamma \gg 1)$ and $n_s = 2(\nu/\gamma^3)(r_0/r_b)^2$ is the field index associated with the beam's self fields. In obtaining (6a, b) we assumed a circular beam cross section and an external field index of 1/2. Note that it is here, in the relation between $\Delta E/E$ and $\Delta\Omega$, that self fields play an important role.

By utilizing the beam envelope equation, the frequency spread term $|\Delta\Omega|_B$ can be expressed in a more illuminating form. The condition for a matched beam, i.e.

non-oscillating minor beam radius, is[12]

$$\frac{1}{4}b^2 = n_s - 1/2 + (r_0\epsilon_n)^2/(r_b{}^2\gamma)^2, \tag{7}$$

where ϵ_n is the normalized transverse beam emittance as measured in the Larmor frame. Substituting (7) into (6b) gives

$$|\Delta\Omega|_B = \frac{\omega_c}{\gamma^2}|(\epsilon_n/r_b)^2/2 - \frac{5}{8}(r_b/r_0)^2\gamma^2 + \nu/2\gamma|. \tag{8}$$

In (8), the first term in brackets is the familiar longitudinal energy spread due to emittance, the second term is a toroidal correction to the first, while the last term is the energy spread associated with the electrostatic potential drop across the beam. Both contributions to the total frequency spread, $|\Delta\Omega|_{\Delta E}$ and $|\Delta\Omega|_B$, are proportional to the various energy spreads. The two different proportionality factors, $|\alpha|$ in the case of $|\Delta\Omega|_{\Delta E}$ and γ^{-2} in the case of $|\Delta\Omega|_B$, arise because the intrinsic particle energy spread produces a revolution-frequency spread primarily by changing the particle's radial position whereas the various energy spreads contributing to $|\Delta\Omega|_B$ merely result in a longitudinal velocity spread. Hence the various contributions to the longitudinal energy spread contribute differently to the frequency spread.

The desired stability criterion is obtained by substituting (6) into (5) and becomes

$$\Gamma \leqslant l\omega_c\left[\frac{1}{2}|\alpha|(\Delta E/E) + |(\epsilon_n/r_b)^2/2 - \frac{5}{8}(r_b/r_0)^2\gamma^2 + \nu/2\gamma|/\gamma^2\right]. \tag{9}$$

Given the intrinsic energy spread, beam radius and emittance, the criterion in (9) implies a limiting beam current which if exceeded will result in instability. As a simple illustration, we will first consider the negative-mass mode in a low current conventional betatron, i.e., $B_\theta = 0$, $n_s \ll 1/2$. From (1), the dispersion relation for the negative mass instability is $\Delta\tilde{\omega}_l{}^2 = -\omega_\parallel{}^2/\omega_\perp{}^2 \approx 4il^2(\nu/\gamma)(Z_\parallel(\omega)/lZ_0)$, where we have assumed that $|\Delta\tilde{\omega}_l| \ll |\tilde{\omega}_\perp| \approx 1/\sqrt{2}$. Approximating $Z_\parallel(\omega)$ by $Z_\parallel(l\omega_c)$ we find that the growth rate is $\Gamma = 2l\omega_c(\nu/\gamma)^{1/2}\,|(Z_\parallel(l\omega_c))|^{1/2}/(lZ_0)^{1/2}$. Using (9) and neglecting the betatron oscillations we recover the well known negative mass stability condition

$$\frac{\nu}{\gamma} \leqslant \frac{lZ_0}{|Z_\parallel(l\omega_c)|}\left(\frac{\Delta E}{E}\right)^2. \tag{10}$$

In obtaining (10) we have assumed that the real and imaginary parts of $Z_\parallel$ are approximately equal.

Next we consider the full dispersion relation (1) for arbitrary b. Here and below we will continue to neglect the effect of betatron oscillations on the stability of the beam. The effect is generally small and, due to the b^2 dependence in (6b), it favors the modified betatron. Therefore, neglect of the betatron oscillations is conservative when comparing the modified and conventional betatrons.

If we may continue to approximate $Z_{\parallel,r}(\omega)$ by $Z_{\parallel,r}(l\omega_c)$ and if we set $\delta = 0$ then (1) is in general a sixth order (fourth order, if $b = 0$) polynomial. By neglecting resistive wall effects we are assuming that these are negligible for the negative mass branch of the dispersion relation in which we will be interested here. Of the six roots, two pairs correspond to transverse modes (a pair being a forward and backward wave, essentially) and one pair to the longitudinal mode.

For a given set of parameters, we have found the roots numerically, for a large range of l; a maximum value of Γ/l is then found and substituted into the stability criterion from which a value of $\Delta E/E$ is computed. An important point to note is that the toroidal field has more of a stabilizing effect on the high l modes than on the low l modes; consequently, as b is increased the behavior of the impedance for *small* l tends to determine the maximum Γ/l and therefore the limiting current.

Typical results are illustrated in Figs. 2 and 3. Here we plot the beam current vs the Lorentzian full width energy spread required for stable motion for $b = 0, 5, 10, 20$, $r_0/a = 6$, $a/r_b = 3$, $R_0/Z_0 = 4$, and $Q = 10$. Figures 2 and 3 are plotted for $\gamma = 3$ and 6 respectively. The dashed lines are plots of

$$\frac{\Delta E}{E} = |1 - 2n_s|\frac{r_b}{r_0}. \tag{11}$$

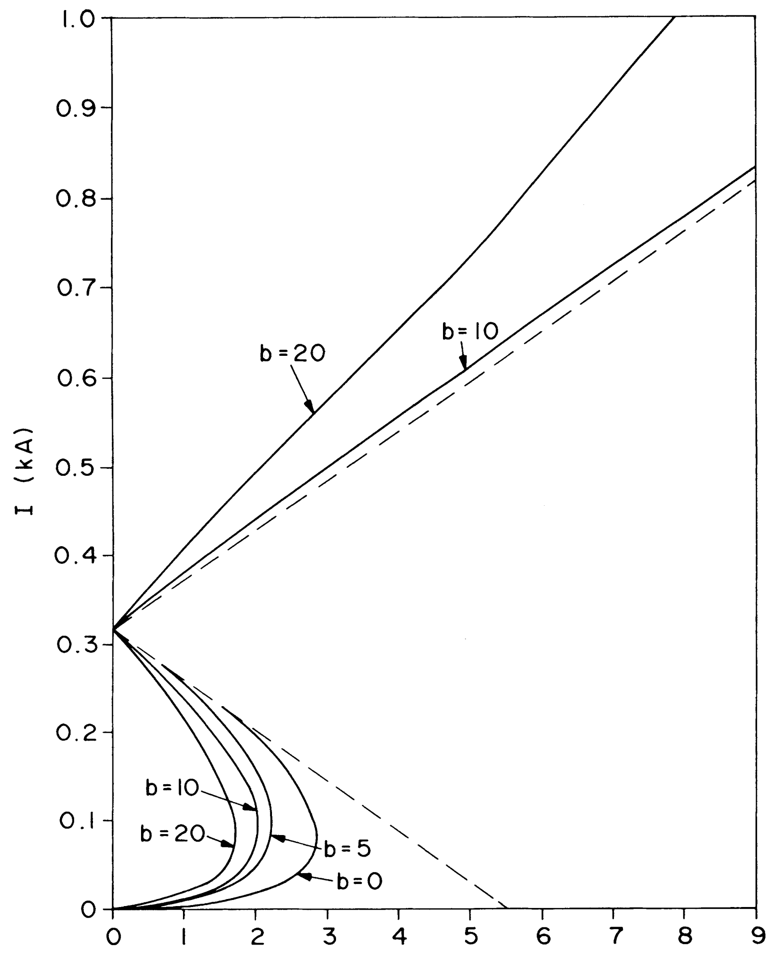

FIGURE 2 Instability threshold current vs energy spread for $b = 0$, 5, and 10. The beam and chamber parameters are $r_0/a = 6$, $a/r_b = 3$, $\gamma = 3$, $R_0/Z_0 = 4$, $Q = 10$.

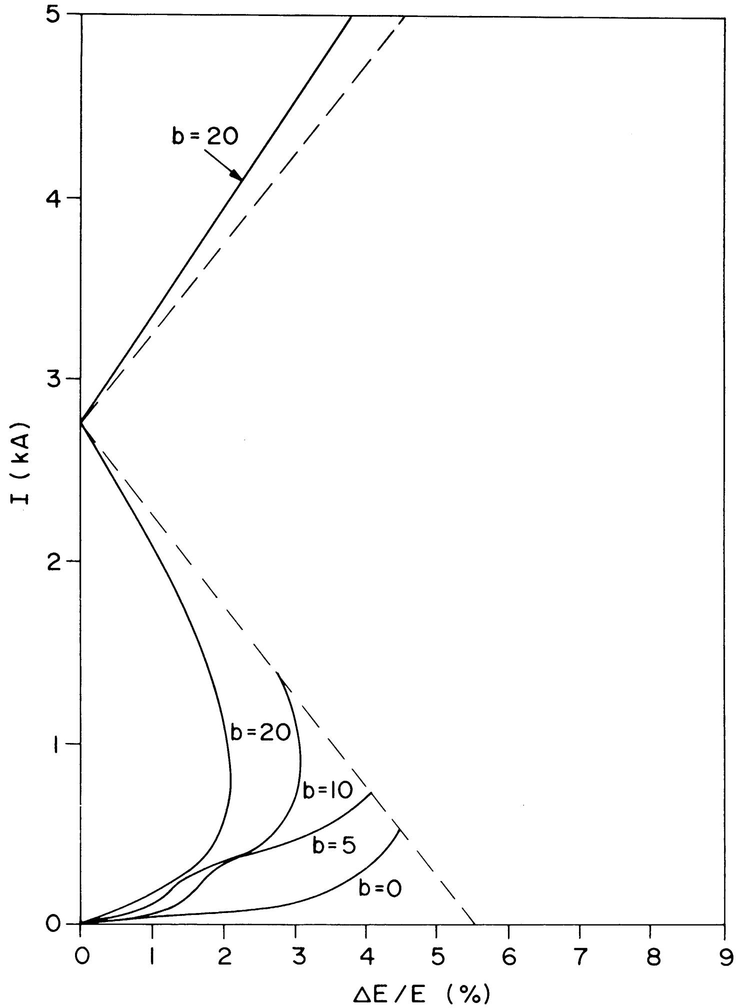

FIGURE 3 Same as Fig. 2 except $\gamma = 6$.

For consistency we must be to the left of these lines. This restriction may be understood by considering the displacement of a particle from the center of the beam

$$\frac{\delta r}{r_0} = \frac{\delta E/E}{\frac{1}{2} - n_s} + \text{betatron oscillations}$$

where δE is the difference between the particular particle energy and the average energy of the particles in the beam. It follows that

$$\frac{r_b}{r_0} \geqslant \left| \frac{\Delta E/E}{1 - 2n_s} \right|, \tag{12}$$

where ΔE is the full width of the distribution.

The effects of the self fields, as represented by n_s in (6a) and (11) are immediately evident in the plots of Figs. 2 and 3. If n_s were zero then the dashed boundary lines would be a single vertical line and the curves would be monotonic. As it is, the effect of the self- fields is basically traceable to the increasing (as n_s increases toward 1/2) then decreasing (as n_s increases beyond 1/2) factor $|\alpha|$. The multi-valuedness of the curves may then be understood as follows (Refer to Fig. 4.): For very small currents (Branch I, in Fig. 4) the cold beam growth rate is small and an increasing function of current. The self-field index, n_s, is negligible compared with 1/2 and the energy spread required for stability is an increasing function of current. This is the regime in which virtually all conventional accelerators operate. As the current is further increased, however, a second branch becomes accessible, shown as Branch II, in Fig. 4: While the cold beam growth rate continues to increase with increasing current, the intrinsic energy-spread

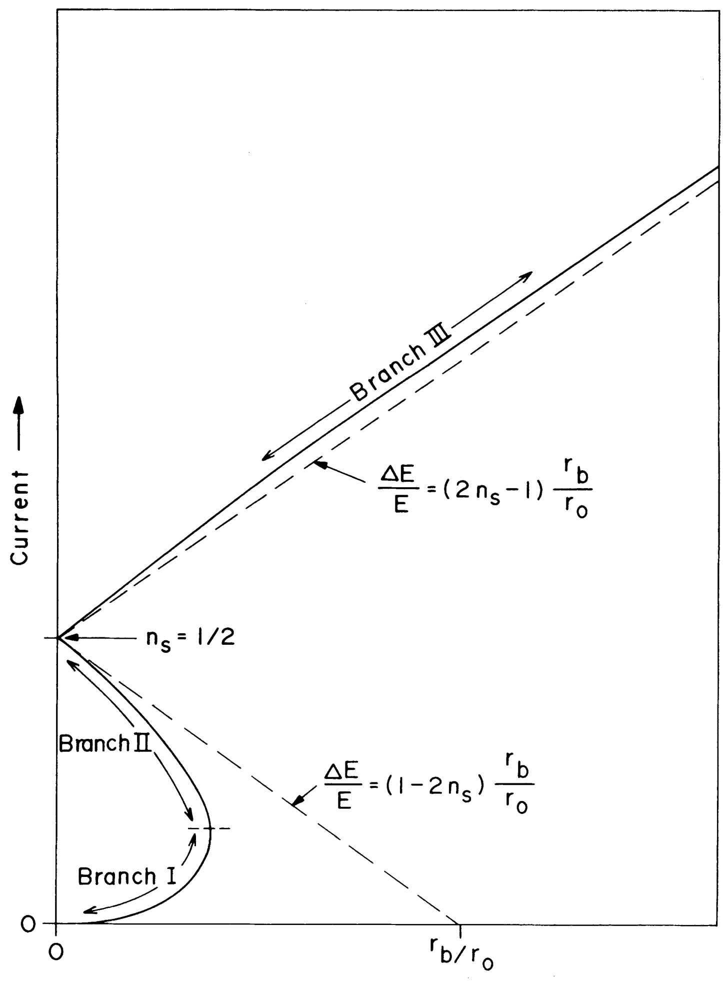

FIGURE 4 Sample plot of limiting current vs energy spread, illustrating the three possible branches.

stabilization mechanism becomes more effective due to the increase in $|\alpha|$ until, near $n_s \lesssim 1/2$ a very small energy spread results in a large spread in angular velocity; the instability is therefore easily stabilized. More simply said, we have stability on the low-current branch (Branch I) because the growth rate is small and on the high-current branch (Branch II) because the stability mechanism is strong.

There is a third branch which appears in the example of Fig. 2 above $n_s = 1/2$ ($I > 316\ A$) for $b = 10$ and 20 and is illustrated as Branch III in Fig. 4. This region is accessible only in the modified betatron since we are constrained in the conventional betatron by the equilibrium condition $n_s < 1/2$. For $b = 5$ in the example shown in Fig. 2 the stable points fall to the right of the dashed line and so are not shown. As the current is increased beyond 316 A, the growth rate increases and the stability mechanism becomes less effective as $|\alpha|$ decreases. Consequently as the current is increased beyond this point the required energy spread increases monotonically. This is illustrated by the third branch in the figure. (The sharp corner between branches II and III of Fig. 4 will probably be rounded off by nonlinear terms in the equations of motion; such terms will become important for $n_s \approx 1/2$ since this condition corresponds to the vanishing of linear restoring forces on the beam particles.)

Finally we comment on the accessibility of the various branches available for small energy spreads. If beam injection proceeds slowly, over many growth times, say, it appears that only the lowest branch is accessible; attempting to add more current will drive the system unstable. If, however, current can be introduced into the accelerator more rapidly, the higher current branches may become accessible. Only a carefully designed experiment can test this speculation. For $b = 5$ and 10 for the parameters of Fig. 2, typical growth times are of the order of 3 particle circulation periods, so that high-current injection on this time scale is a practical experimental possibility. The third branch in Fig. 2 is clearly the most promising for very high current operation.

V. CONCLUSIONS

We have shown that the addition of a toroidal magnetic field to a conventional betatron may significantly improve the current-carrying capacity of the betatron by controlling the collective instabilities that limit the current. The calculation has included self-field effects and a simple, though realistic model for the longitudinal and transverse impedances. The stabilizing effects of betatron oscillations, which have been shown to include the effects of emittance, toroidal geometry, and energy shear due to the electrostatic potential drop across the beam, become stronger as the toroidal field is increased, given a fixed beam radius. Inclusion of self-field effects in the stability criterion (6) has been shown to lead to a multi-valuedness in the current vs energy spread plot which has been interpreted as the result of the competition between the growth and stabilization mechanisms. Accessibility of the high-current branches may depend on the duration of the injection process.

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REFERENCES

1. A. G. Bonch-Osmolovskii, G. V. Dolbilov, I. N. Ivanov, E. A. Perelstein, V. P. Sarantsev, and O. I. Yarkova; JINR Report P9-4135 Dubna (USSR), 1968 (Unpublished).
2. P. Sprangle and C. A. Kapetanakos, *J. Appl. Phys.*, **49**, 1 (1978).
3. N. Rostoker, *Comments on Plasma Phys.*, **6**, 91 (1980).
4. D. Chernin and P. Sprangle, *Particle Accelerators*, **12**, 85 (1982).
5. G. Barak and N. Rostoker, *Phys. Fluids*, **26**, 856 (1983).
6. V. K. Neil and A. M. Sessler, *Rev. Sci. Instr.*, **36**, 429 (1965).
7. L. J. Laslett, V. K. Neil, and A. M. Sessler, *Rev. Sci. Instr.*, **36**, 436 (1965).
8. E. P. Lee, A. Faltens, L. J. Laslett, and L. Smith, *IEEE Trans. Nucl. Sci.*, **30**, 2504 (1983).
9. R. W. Landau, *Phys. Fluids*, **11**, 205 (1968).
10. P. Sprangle and J. L. Vomvoridis, NRL Memorandum Report 4688 (1982).
10a. R. Davidson and H. Uhm, *Phys. Fluids*, **25**, 2089 (1982).
11. C. Pellegrini, in *Physics of High Energy Particle Accelerators* (Fermilab Summer School, 1981), R. A. Carrigan, F. R. Huson, and M. Month, eds. pp. 77ff.
12. See, e.g. J. D. Lawson, *The Physics of Charged Particle Beams* (Oxford University Press, 1977), Eq. (4.37).